

Multiple choice.

1. A
2. C
3. C
4. B
5. D
6. B
7. B
8. A
9. D
10. C

Question 11

(a)

$$i. |y|= \sqrt{5} \quad |x|=5 \quad y \cdot x = 11$$

$$\cos \theta = \frac{u \cdot v}{|u||v|}$$

$$= \frac{11}{\sqrt{5} \cdot 5}$$

$$\therefore \cos \theta = \frac{11\sqrt{5}}{25}$$

∴ Acute angle between y and x is $10^\circ 18'$

1 mark: correct solution.

$$ii. |y|= \sqrt{10} = 3\sqrt{10} \quad |x|= \sqrt{18} = 3\sqrt{2} \quad y \cdot x = 36$$

$$\cos \theta = \frac{y \cdot x}{|y||x|}$$

$$= \frac{36}{3\sqrt{10} \cdot 3\sqrt{2}}$$

$$= \frac{4}{2\sqrt{5}}$$

$$\cos \theta = \frac{2\sqrt{5}}{5}$$

∴ Acute angle between y and x is $26^\circ 34'$

1 mark: correct solution.

(b)

$$\int 3 \cos^2 3x \, dx$$

$$= \frac{3}{2} \int \cos 6x - 1 \, dx$$

$$= \frac{3}{2} \left(\frac{1}{6} \sin 6x - x \right) + C$$

$$= \frac{1}{4} \sin 6x - \frac{3x}{2} + C$$

1 mark: correct trigonometric substitution

1 mark: correct solution

-1 mark if no '+C'

(c)

$$\frac{1}{x+1} < 3$$

$$x+1 < 3x^2 + 6x + 3$$

$$0 < 3x^2 + 5x + 2$$

$$(3x+2)(x+1) > 0$$



$$x < -1 \quad \text{or} \quad x > -\frac{2}{3}$$

1 mark: substantial working leading to solution

1 mark: correct solution

-1 mark if multiply by $(x+1)$

(d)

$$P(x \geq 3) = P(3) + P(4) + P(5)$$

$$P(3) = {}^6C_3 (0.66)^3 (0.34)^2 = 0.33234\dots$$

$$P(4) = {}^6C_4 (0.66)^4 (0.34)^1 = 0.32257\dots$$

$$P(5) = {}^6C_5 (0.66)^5 (0.34)^0 = 0.125233\dots$$

$$P(3) + P(4) + P(5) = 0.7801 \quad (45\%) \quad \text{or} \quad 78.01\%$$

1 mark: identifying / understanding 'majority'

1 mark: correct solution.

(e)

∴ $(x+1)$ and $(x-5)$ are factors, then

-1 and 3 are roots of $P(x)$, meaning

$$P(-1) = 0$$

$$P(3) = 0$$

$$P(-1) = -a + b + 9 - 15 = 0$$

$$a - b = 4 \quad \sim (1)$$

$$P(3) = 27a + 9b - 57 - 15 = 0$$

$$27a + 9b = 72 \quad \sim (2)$$

(1) + (2):

$$4a = 12$$

$$a = 3 \quad \sim \text{sub into (1)}$$

$$3 - b = 4$$

$$-b = 1$$

$$b = -1$$

1 mark: recognising the roots

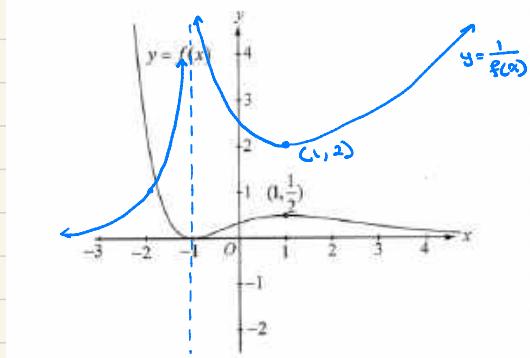
1 mark: substantial working

1 mark: correct solution

Question 11 continued ...

(f)

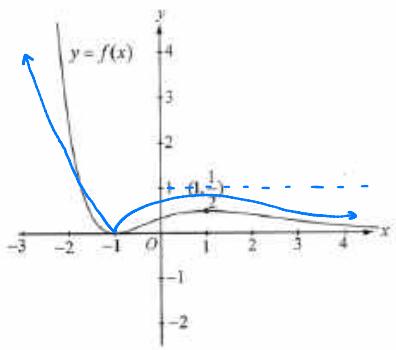
i. $y = \frac{1}{f(x)}$



1 mark: significant points labelled.

1 mark: correct solution.

ii. $y = f(f(x))$



1 mark: significant property at $x = -1$, pointed

1 mark: correct solution

Question 12

(a)

i. SHOW TRUE FOR $n=1$

RTP: $2^n + 1 = 3A$ for odd $n \geq 1 \in \mathbb{Z}^+$, where $A \in \mathbb{Z}^+$

$$\text{LHS} = 2^1 + 1$$

$$= 2^1 + 1$$

$$= 3$$

$$= 3(1)$$

$$= 3A \quad \text{where } A=1$$

$$= \text{RHS}$$

ii. ASSUME TRUE FOR $n=k$

$$2^k + 1 = 3B \quad \text{where } B \in \mathbb{Z}^+$$

iii. PROVE TRUE FOR $n=k+2$

RTP: $2^{k+2} + 1 = 3C$ where $C \in \mathbb{Z}^+$

$$\text{LHS} = 2^{k+2} + 1$$

$$= 4 \cdot 2^k + 1$$

$$= 4(2^k + 1) + 1$$

$$= 4(3B - 1) + 1$$

$$= 12B - 3$$

$$= 3(4B - 1)$$

$$= 3C \quad \text{where } C = 4B - 1$$

$$= \text{RHS}$$

iv. since expression is true for $n=1$, then it

is true for $n=3$, and so it is true for

$n=5$ etc.

thus, $2^n + 1$ is divisible by 3 for all odd $n \in \mathbb{Z}^+$

Proven by mathematical induction.

1 mark: showing true for $n=1$

1 mark: correct assumption

1 mark: correct proof.

(b)

let S be the probability of getting a black marble

$\therefore P$ be probability of green.

$$S = 0.3$$

$$P = 0.7$$

$$\begin{aligned} P(X \leq 4) &= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) \\ &= {}^{12}C_0 0.3^0 0.7^12 + {}^{12}C_1 0.3^1 0.7^11 + {}^{12}C_2 0.3^2 0.7^10 + \\ &\quad {}^{12}C_3 0.3^3 0.7^9 + {}^{12}C_4 0.3^4 0.7^8. \\ &= 0.7237 \quad (\text{Ans}). \end{aligned}$$

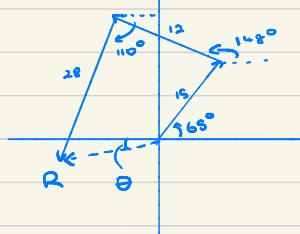
1 mark: correct probabilities for Black and Green marbles

1 mark: substantial working leading to solution

1 mark: correct solution.

(c)

Resolve components into x and y



x -component:

$$15 \cos 65 + 12 \cos 148 + 28 \cos(-110) \approx -13.414.$$

negative direction.

y -component:

$$15 \sin 65 + 12 \sin 148 + 28 \sin(-110) \approx -6.358$$

$$|\vec{QR}| = \sqrt{6.358^2 + 13.414^2} = 14.84 \text{ km}$$

$$\theta = \tan^{-1}\left(\frac{-6.358}{-13.414}\right) = 25^\circ 23'$$

Boat is 14.84 km from the pier at a bearing of $864^\circ 38' \text{W}$

1 mark: correct bearing

1 mark: correct distance

1 mark: substantial working leading to solution

(d)

$$\text{i. } \frac{dy}{dx} = e^{x+y}$$

$$\int e^{-y} dy = \int e^x dx$$

$$-\frac{1}{e^y} = e^x + C.$$

$$-1 = e^{x+y} + Ce^y$$

$$e^{x+y} + Ce^y + 1 = 0.$$

1 mark: correct solution.

$$\text{ii. } \frac{dy}{dx} = x^2(1+y^2)$$

$$\int \frac{dy}{1+y^2} = \int x^2 dx$$

$$\tan^{-1} y = \frac{x^3}{3} + C$$

$$y = \tan\left(\frac{x^3}{3} + C\right)$$

1 mark: correct solution.

Question 12 continued...

$$\text{iii. } \int \frac{dy}{y^2} = \int 3x^2 dx$$

$$-\frac{1}{y} = x^3 + C$$

$$y=1, x=0$$

$$-1 = C$$

$$-\frac{1}{y} = x^3 - 1$$

$$y = \frac{1}{1-x^3}$$

1 mark: correct solution.

(e)

$$\text{LHS: } (0) + 2(1) + 3(2) + \dots + (n+1)(n) = 2^n \left(\frac{n}{2} + 1\right)$$

consider;

$$\begin{aligned} x(1+x)^n &= x((0) + (1)x + (2)x^2 + \dots + (n)x^n) \\ &= (0)x + (1)x^2 + (2)x^3 + \dots + (n)x^{n+1} \end{aligned}$$

differentiate both sides -

$$\text{LHS} = (1+x)^n + n(1+x)^{n-1} \cdot x$$

$$\text{RHS} = (0) + 2(1)x + 3(2)x^2 + \dots + (n+1)(n)x^n$$

sub $x=1$

$$\begin{aligned} \text{LHS} &= 2^n + n(2)^{n-1} \\ &= 2^n \left(1 + \frac{n}{2}\right) \end{aligned}$$

$$\text{RHS} = (0) + 2(1) + 3(2) + \dots + (n+1)(n).$$

Hence,

$$\text{RHS} = \text{LHS}$$

$$(0) + 2(1) + 3(2) + \dots + (n+1)(n) = 2^n \left(\frac{n}{2} + 1\right)$$

as required.

1 mark: correct expansion of $x(1+x)^n$

1 mark: differentiating

1 mark: correct solution.

Question 13

(a)

$$\sec^2 x + \tan x - 7 = 0$$

$$\tan^2 x + \tan x - 6 = 0$$

$$(\tan x + 3)(\tan x - 2) = 0$$

$$\tan x = -3 \quad \tan x = 2$$

\therefore acute principle angles

$$71^\circ 34' \text{ and } 63^\circ 26'$$

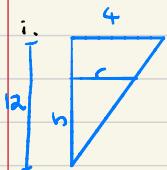
$$\therefore x = 108^\circ 26', 288^\circ 26', 63^\circ 26', 243^\circ 26'$$

1 mark: uses $\sec^2 x = \tan^2 x + 1$

1 mark: correct principle angles

1 mark: correct solutions

(b)



$$\frac{h}{r} = \frac{c}{4}$$

$$r = \frac{1}{3}h \text{ as required.}$$

1 mark: correct solution.

$$\text{ii. Find } \frac{dv}{dt} \text{ when } h=9$$

$$V = \frac{1}{3} \pi r^2 h$$

$$= \frac{1}{3} \pi \left(\frac{h}{3}\right)^2 h$$

$$V = \frac{\pi h^3}{27}$$

$$\text{Given } \frac{dv}{dt} = 3$$

$$\frac{dv}{dh} = \frac{3\pi h^2}{27} = \frac{\pi h^2}{9}$$

$$\frac{dv}{dt} = \frac{dv}{dh} \cdot \frac{dh}{dt}$$

$$3 = \frac{\pi 9^2}{9} \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{3}{9\pi} = \frac{1}{3\pi} \text{ cm s}^{-1}$$

1 mark: Volume in terms of h.

1 mark: applies chain rule correctly

1 mark: correct solution.

(c)

$$\text{i. } \tan \alpha = \frac{h}{3h}$$

$$\alpha = \tan^{-1} \frac{1}{3}$$

$$\therefore \alpha = 18^\circ 26'$$

1 mark: correct solution

$$\text{ii. } (h+300)^2 = (3h)^2 + (4h)^2 - 2 \cdot 3 \cdot 4 h^2 \cos 60^\circ$$

$$13h^2 = h^2 + 600h + 90000$$

$$0 = 12h^2 - 600h - 90000$$

$$h^2 - 50h - 7500 = 0$$

$$h = \frac{50 + \sqrt{625 - 4 \times 1 \times -7500}}{2}$$

$$h = 115.1 \text{ m} \quad (h > 0)$$

1 mark: correct initial equation (i).

1 mark: correct simplification of equation

1 mark: correct solution.

(d)

$$\text{let } \vec{OA} = \vec{a} \text{ and } \vec{OB} = \vec{b}$$

$$\vec{OD} = \lambda \vec{OA} = \lambda \vec{a}$$

$$\vec{OE} = \lambda \vec{OB} = \lambda \vec{b}$$

$$\text{now, } \vec{AB} = \vec{b} - \vec{a}$$

$$\vec{DE} = \vec{a} - \vec{b}$$

$$= \lambda \vec{b} - \lambda \vec{a}$$

$$= \lambda (\vec{b} - \vec{a})$$

$$\vec{DE} = \lambda (\vec{AB})$$

since \vec{DE} is a scalar multiple of \vec{AB} , $DE \parallel AB$ and reading off answer, DE is λ times AB.

1 mark: correct expression of \vec{a} and \vec{b}

1 mark: correct solution

(e)

$$\text{n.o. arrangements} = \frac{11!}{2! 2! 2!}$$

$$= 4989600$$

1 mark: accounts for double letters

1 mark: correct solution.

Question 14.

(a)

i. SHOW TRUE FOR $n=1$

$2^1 \times 2^1$



can be formed with an L-tile if one square is removed.

ii. ASSUME TRUE FOR $n=k$ $2^k \times 2^k$



\rightarrow Assume it is $2^k \times 2^k$

iii. PROVE TRUE FOR $n=k+1$

$$\text{to get } 2^{k+1} \times 2^{k+1} = 2 \cdot 2^k \times 2 \cdot 2^k$$



The centre with 4 blocks removed can be filled with an L-tile, leaving one block removed.

\therefore True for $n=k+1$.

iv. Thus proven true by mathematical induction

(b)

t = time (hrs)

$T(t)$ = Temperature at time t ($^{\circ}\text{C}$)

It is a first order ODE

$$\frac{dT}{dt} = -k$$

↑
Decrease in temperature.

\therefore mathematically

$$\frac{dT}{dt} = -k(T-20) \quad \text{where ambient } T=20^{\circ}\text{C} \quad P \\ \text{given}$$

Solving ODE, inspection

$$\ln|T-20| = -kt + C$$

let $t=0$ be at 11:30am, $T=24.5^{\circ}\text{C}$

$$\ln|4.5| = C$$

$$-kt = \ln\left|\frac{T-20}{4.5}\right|$$

now at $t=1$ (12:30pm), $T=24$

$$-k = \ln\left|\frac{4}{45}\right|$$

$$t = \frac{\ln\left|\frac{T-20}{4.5}\right|}{\ln\left|\frac{4}{45}\right|}$$

time of death, $T=36.5^{\circ}$

$$t = \frac{\ln\left|\frac{16.5}{4.5}\right|}{\ln\left|\frac{4}{45}\right|}$$

$= 11 \text{ hrs } 2 \text{ mins (nearest min)}$

$\therefore 11 \text{ hrs } 2 \text{ mins before } 11:30 \text{ am.}$

\therefore Time of death = 12:28am (nearest min).

1 mark: Identifying ODE

1 mark: correct solution for ODE / IVP

1 mark: Assigns suitable time for $t=0$

1 mark: correct solution.

(c)

Given light passing through is proportional to thickness of slab.

$$\frac{dx}{db} = -kx$$

Let b be thickness and $x=x(b)$ be fraction of light passing through slab.

at $b=1 \text{ cm}$, if $\frac{1}{4}$ is absorbed, then $x=\frac{3}{4}$

Also worth noting: $x(0)=1$ (100% light passes)

Now,

$$\frac{dx}{db} = -kx$$

$$\ln|x| = -kb + C$$

$$b=0, x=1$$

$$x = e^{-kb}$$

Finding k . $b=1, x=\frac{3}{4}$

$$\frac{3}{4} = e^{-k}$$

$$-k = \ln\left(\frac{3}{4}\right)$$

$$\therefore x = e^{-b\ln\left(\frac{3}{4}\right)}$$

Now we want to find when 1% absorbed

so 99% passes.

$$0.99 = e^{b\ln\left(\frac{3}{4}\right)}$$

$$\ln 0.99 = b\ln\left(\frac{3}{4}\right)$$

$$b = 0.035 \text{ cm (3dp).}$$

1 mark: Identify ODE

1 mark: Solve IVP

1 mark: correct solution

(d)

At least two: $P(X \geq 2)$

where $X =$ number of people with same birthday.

$$= P(2) + P(3) + \dots + P(25)$$

easier to consider the complement.

$$P(X \geq 2) = 1 - P(X < 2), \text{ or}$$

$$P(A) = 1 - P(A^c)$$

$A^c \rightarrow$ no two people share the same birthday.

Total number possibilities = 365^{25} (permutation)

$$A^c = \frac{365!}{(365-25)!} = \frac{365!}{340!}$$

Hence;

$$P(A^c) = \frac{365!}{340! \cdot 365^{25}}$$

$$\approx 0.4313.$$

now,

$$P(A) = 1 - P(A^c)$$

$$= 1 - 0.4313$$

$$= 0.5687 \text{ or } 56.87\%.$$

1 mark: correct complement

1 mark: correct combinatoric solution

1 mark: substantial working leading to solution

1 mark: correct solution