PΣΛK

4U MOCK EXAM

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions	• Reading time – 5 minutes • Working time – 3 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • In Questions 11–16, show relevant mathematical reasoning and/or calculations
Total marks: 100	**Section I – 10 marks** (pages 2–6) • Attempt Questions 1–10 • Allow about 15 minutes for this section **Section II – 90 marks** (pages 7–17) • Attempt Questions 11–16 • Allow about 2 hours and 45 minutes for this section

Section I

10 Marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Q1-10

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1 What is the value of $(6+7i)^2$

A. $-13+84i$

B. $-14+72i$

C. $13-84i$

D. $42\sqrt{3}+21i$

2 Which of the following is a primitive of $\frac{x^3}{x^2+1}$

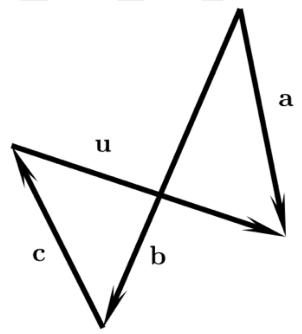
A. $x^3 + x\ln\left|\frac{x+1}{x-1}\right| + 2$

B. $\frac{1}{2}x^3 - \frac{1}{2}\ln|x^2+1| + 5$

C. $\frac{1}{2}x^2 - \frac{1}{3}\ln|x^2+1| + 7$

D. $\frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + 3$

3 Express the vector $\boldsymbol{u}$ in terms of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$



A. $-a+b+c$

B. $a-b-c$

C. $a+b+c$

D. $-a-b-c$

4 Which of these integrals has the largest value?

A. $\int_0^{\frac{\pi}{4}} \tan x \, dx$

B. $\int_0^{\frac{\pi}{4}} \tan^2 x \, dx$

C. $\int_0^{\frac{\pi}{4}} 1 - \tan x \, dx$

D. $\int_0^{\frac{\pi}{4}} 1 - \tan^2 x \, dx$

5 Simplify i^{2020}

A. i

B. -1

C. 1

D. $-i$

6 Which expression is equal to $\int \frac{dx}{\sqrt{7-6x-x^2}}$

A. $\sin^{-1}\left(\frac{x+3}{2}\right) + c$

B. $\sin^{-1}\left(\frac{x+3}{4}\right) + c$

C. $\sin^{-1}\left(\frac{x-3}{2}\right) + c$

D. $\sin^{-1}\left(\frac{x-3}{4}\right) + c$

7 Which is the logical equivalent statement to

"If I am in Mathematics class today, then I am in school"

A. "If I am in school today, then I am in Mathematics class"

B. "If I am not in school today, then I am not in Mathematics class"

C. "If I am not in Mathematics class, then I am not in school today"

D. "If I am in school today, then I am not in Mathematics class"

8 Consider $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x}{(1+\tan x)^2} dx$

After an appropriate substitution, which of the following is equivalent to I?

A. $\int_0^2 \frac{1}{u^2} du$

B. $\int_0^2 \frac{1}{(1+u)^3} du$

C. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{u} du$

D. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{u^3} du$

9 A particle is dropped vertically in a medium where the resistance is $\frac{k}{v^3}$. If down is taken to be the positive direction, the terminal velocity is

A. $\sqrt[3]{\frac{mg}{k}}$

B. $\sqrt[3]{\frac{mk}{g}}$

C. $\sqrt[3]{\frac{k}{mg}}$

D. $\sqrt[3]{\frac{gk}{m}}$

10 $\frac{6}{2x^2-5x+2}$ expressed as a sum of partial fractions is

A. $\frac{2}{2x-1} - \frac{4}{x-2}$

B. $\frac{2}{x-2} - \frac{4}{2x-1}$

C. $\frac{2}{2x-1} - \frac{4}{x+2}$

D. $\frac{4}{2x-1} - \frac{2}{x-2}$

End of Section I

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11-16, your response should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Let $z = 4 + 6i$ and $w = 5 + i$

(i) Find $z + \overline{w}$ **1**

(ii) Express $\frac{z}{w}$ in the form $x + iy$, where x and y are real numbers **2**

(b) Find $\int \frac{x^3}{\sqrt{1+x^2}} dx$ **2**

(c) $w = \frac{-9+3i}{1-2i}$

(i) State the modulus of w **1**

(ii) State the argument of w **1**

(iii) Write w in mod-arg form **1**

(d) By using the substitution $w = z^2 - z$ or otherwise, find in exact form the four solutions of $z^4 - 2z^3 - 2z^2 + 3z - 4 = 0 \quad z \in \mathbb{C}$ **4**

(e) $2z^2 - (3 + 8i)z - (m + 4i) = 0 \qquad z \in \mathbb{C}$

Given that m is a real constant, find two solutions of the above equation given further that one of these solutions is real. **3**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

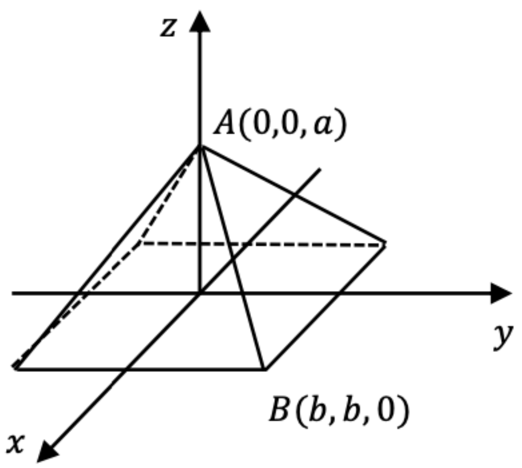
(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos 2x}{2-\sin 2x}\,dx$ **3**

(b) Find the sum of

$$\cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta$$ **3**

(c) A rectangular prism with side lengths $6, 8\ and\ 10$ units has both ends of one of its longest diagonals along the $x - axis$. Prove that all points on the surface of the prism satisfy $|z| \le 5\sqrt{2}$ **3**

(d) A square based pyramid has its base on the $x - y$ plane, with its apex at $A(0, 0, a)$. The four triangles forming its sides are isosceles with sides in the ratio $2: 2: 1$, the short side being the bottom side. One of the four vertices of the square base is $B(b, b, 0)$, where $b > 0$. Find b in terms of a.



2

(e) Solve the quadratic equation

$$iz^2 - 2\sqrt{2}z - 2\sqrt{3} = 0 \qquad z \in \mathbb{C}$$ **4**

Give answers in the form of $x + iy$, where x and y are exact real constants.

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Prove for integer x, x^2 is divisible by 9 if and only if x is a multiple of 3. **2**

(b) Prove the following statement is false

$$|2x+5| \leq 9 \Longrightarrow |x| \leq 4$$ **2**

(c) If $a > b > 0$ for real a and b, prove that $2^{-a^2} < 2^{-b^2}$ **2**

(d) Prove by induction for $n \geq 2$ that

$$1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \cdots + n^3$$ **4**

(e) Evaluate $\int_{-2}^{2}(x+x^3+x^5)(1+x^2+x^4)\,dx$ **2**

(f) Find $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) A cube has opposite vertices at the origin and $(2, 2, 2)$. State the equation of the four diagonals. Are the diagonals perpendicular? **4**

(b) A triangle has vertices $A(0,0,0)$, $B(0,2,4)$ and $C(4,2,0)$. Find the equations of the three medians and show that they are concurrent. **3**

(c) A 20kg trolley is pushed with a force of 100 N. Friction causes a resistive force which is proportional to the square of the trolley's velocity.

(i) Show that $\ddot{x} = 5 - \frac{kv^2}{20}$ where k is a positive constant. **2**

(ii) If the trolley is initially stationary at the origin, show that the distance travelled when its speed is V is given by

$$x = \frac{10}{k}\ln\left(\frac{100}{100-kV^2}\right)$$ **2**

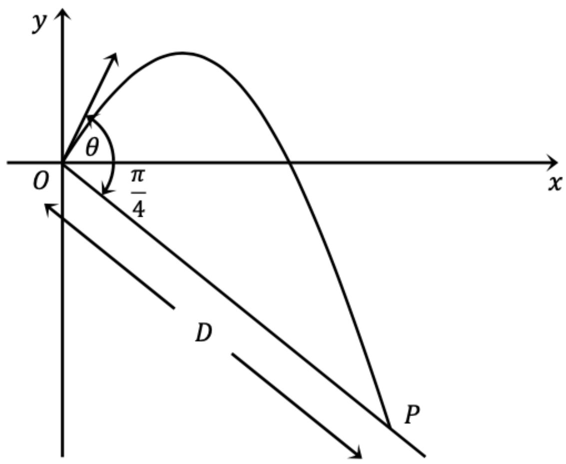
(d) Prove the Harmonic Mean ≤ the Geometric Mean ≤ the Arithmetic Mean ≤ the Quadratic Mean for two numbers, mathematically,

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$ **4**

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) The take-off point O on a ski jump is located at the top of a downslope. The angle between the downslope and the horizontal is $\frac{\pi}{4}$. A skier takes off from O with velocity Vms^{-1} at angle θ to the horizontal, where $0 \leq \theta < \frac{\pi}{2}$. The skier lands on the downslope at some point P a distance of D metres from O.



The flight path of the skier is given by $x = Vt\cos\theta$, $y = -\frac{1}{2}gt^2 + Vt\sin\theta$, where t is the time in seconds after take-off. (Do NOT prove this)

(i) Show that the cartesian equation of the flight path of the skier is given by

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$ **2**

(ii) Show that $D = 2\sqrt{2}\frac{V^2}{g}\cos\theta\,(\cos\theta + \sin\theta)$ **3**

(iii) Show that $\frac{dD}{d\theta} = 2\sqrt{2}\frac{V^2}{g}(\cos 2\theta - \sin 2\theta)$ **3**

(iv) Show that D has a maximum value and find the value of θ for which this occurs.

Question 15 continues on the next page

(b) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ for all integers $n \geq 0$

(i) Show that $I_n = \frac{1}{n-1} - I_{n-2}$ for integers $n \geq 2$ **3**

(ii) Hence find $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$ **2**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

(a) You have three pegs and a collection of disks of different sizes. Initially all of the disks are stacked on top of each other according to size on the first peg, i.e. the largest disk being on the bottom and the smallest on top. A move in this game consists of moving a disk from one peg to another, subject to the condition that a larger disk may never rest on a smaller one. The objective of the game is to find a number of permissible moves that will transfer all of the disks from the first peg to the third peg, making sure that the disks are assembled on the third peg according to size. The second peg is used as an intermediate peg. Prove that it takes $2^n - 1$ moves to move n disks from the first peg to the third peg. **4**

(b) A particle is moving in a medium where resistance to motion is proportional to the square of velocity, so $R = -kv^2$. At some point in its flight $\dot{x} = 7ms^{-1}$ and $\dot{y} = 24\, ms^{-1}$.

(i) Use similar triangles to find the horizontal and vertical components of resistance at the point, and prove that the total resistance and its components satisfy Pythagoras' Theorem. **3**

(ii) Show that the horizontal and vertical components at the point can be found using $R_x = -kv\dot{x}$ and $R_y = -kv\dot{y}$ **2**

(c) A body of unit mass is projected vertically upwards in a medium that has a constant gravitational force g and a resistance $\frac{v}{10}$, where v is the velocity of the projectile at a given time t. The initial velocity is $10(20 - g)$

(i) Show that the equation of motion fo the projectile is $\frac{dv}{dt} = -g - \frac{v}{10}$ **1**

(ii) Show that the time T for the particle to reach its greatest height is given by $T = 10\ln\left(\frac{20}{g}\right)$ **2**

(iii) Show that the maximum height H is given by
$H = 2000 - 10g[10 + T]$ **2**

(iv) If the particle the falls from this height, find the terminal velocity in this medium. **1**

End of paper

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2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

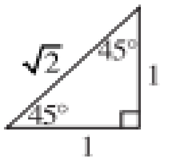
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

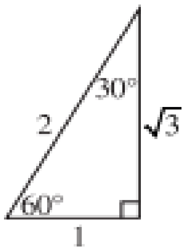


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

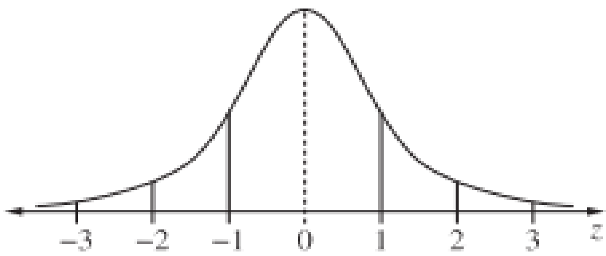
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \text{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\text{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2} + y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^{2}(x - c)$$