# MC and 11

Tuesday, 15 September 2020 12:39 am

multiple choice.

1. A
2. D
3. A
4. D
5. C
6. B
7. B
8. A
9. C
10. B

short answer.

Question 11

(a) i.

$z = 4 + 6i$

$\bar{w} = 5 - i$

$\therefore z + \bar{w} = 4 + 6i + 5 - i$

$= \boxed{9 + 5i}$ ① correct answer.

ii.

$\frac{z}{w} = \frac{4+6i}{5+i} \times \frac{5-i}{5-i}$ ① correct conjugate

$= \frac{20 - 4i + 30i - 6i^2}{26}$

$= \frac{26 + 26i}{26}$

$= 1 + i$ ① correct answer.

(b)

let $u = 1 + x^2 \Rightarrow x^2 = u - 1$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = x\,dx$ ① A possible substitution.

$I = \int \frac{u-1}{\sqrt{u}}\,du$

$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$

$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}$

$I = \frac{2}{3}(1+x^2)^{\frac{3}{2}} - 2\sqrt{1+x^2} + C$ ① correct answer - wrong if no $+C$.

(c) $w = \frac{-9+3i}{1-2i} \times \frac{1+2i}{1+2i}$

$= \frac{-9 - 18i + 3i + 6i^2}{5}$

$= \frac{-15}{5} - \frac{15i}{5}$

$= -3 - 3i$

i. $3\sqrt{2}$ ① correct answer

ii. $-\frac{3\pi}{4}$ ① correct answer

iii. $3\sqrt{2}\,\text{cis}\,-\frac{3\pi}{4}$ ① correct answer.

# 11 continued

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(a) $w = z^2 - z \Rightarrow w^2 = z^4 - 2z^3 + z^2$ ① squaring substitution.

Hence,

$z^4 - 2z^3 - 2z^2 + 3z - 4 = 0$

$(z^4 - 2z^3 + z^2) - 3z^2 + 3z - 4 = 0$

$(z^2 - z)^2 - 3(z^2 - z) - 4 = 0$ ① identifying the quadratic and factorising (either by inspection or by quadratic equation).

$(z^2 - z + 1)(z^2 - z - 4) = 0$

$\therefore z^2 - z = -1$ or $z^2 - z = 4$

we can complete the square (I'll do a variation of completing the square).

$4z^2 - 4z = -4$ | $4z^2 - 4z = 16$

$4z^2 - 4z + 1 = -3$ | $4z^2 - 4z + 1 = 17$

$(2z-1)^2 = -3$ | $(2z-1)^2 = 17$

$2z - 1 = \pm i\sqrt{3}$ | $2z - 1 = \pm\sqrt{17}$

$z = \frac{1 \pm i\sqrt{3}}{2}$ | $z = \frac{1 \pm \sqrt{17}}{2}$

② correct solutions −1m per pair so; if only $z = \frac{1 \pm i\sqrt{3}}{2}$ and not $\frac{1 \pm \sqrt{17}}{2}$; only 1m.

(c) if equation is to have real solution, let $z = x$ be the real solution and $x \in \mathbb{R}$.

then,

$2x^2 - (3+8i)x - (m+4i) = 0$

collect real and imaginary.

$2x^2 - 3x - 8ix - m - 4i = 0.$

$(2x^2 - 3x - m) + i(-8x - 4) = 0$

by equating

$-8x - 4 = 0$

$x = -\frac{1}{2}$ ① one root

then sub into real part

$2(-\frac{1}{2})^2 - 3(-\frac{1}{2}) - m = 0$

$\frac{1}{2} + \frac{3}{2} - m = 0$

$m = 2$ ① finding m

Thus

$2z^2 - (3+8i)z - (2+4i) = 0$

$(2z+1)(z - (2+4i)) = 0$ ← recognise factorisation here, don't need quadratic formula b/c we already have one root → alternative, use sum of roots or product of roots.

$z = -\frac{1}{2}$ or $z = 2 + 4i$ ① other root

12

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Question (12)

(a)

let $u = 2 - \sin 2x$

$$\frac{du}{dx} = -2\cos 2x$$ ① correct substitution

$$\cos 2x \, dx = -\frac{du}{2}$$ ① Either change of parameters or convert $u$ back into $x$ and sub original parameters.

$$\therefore I = \frac{1}{2}\int -\frac{du}{u}$$

$$= -\frac{1}{2}\Big[\ln u\Big]_2^2$$

$$= 0$$ ① correct answer.

(b) $\cos\theta + i\sin\theta = e^{i\theta}$

Consider $S_n = e^{i\theta} + e^{i2\theta} + \dots + e^{in\theta}$ ① Producing the GP

now, GP

$$S_n = \frac{a(r^n - 1)}{r - 1}$$ $e^{i2\theta - i\theta}$

$$= \frac{e^{i\theta}(1 - e^{in\theta})}{1 - e^{i\theta}}$$

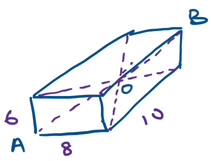
now, $1 - e^{i\theta} = e^{i\frac{\theta}{2}}(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}) = -2ie^{i\frac{\theta}{2}}\sin(\frac{\theta}{2})$ ① recognising trick

$$\therefore S_n = e^{i\theta}\left(\frac{e^{in\frac{\theta}{2}}\sin(\frac{n\theta}{2})}{e^{i\frac{\theta}{2}}\sin(\frac{\theta}{2})}\right) = e^{i(n+1)\frac{\theta}{2}}\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

now:

$$\sum_{k=1}^{n}\cos(k\theta) = Re(S_n) = \cos\left(\frac{(n+1)\theta}{2}\right)\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}.$$ ① correct answer.

(c) Consider.




all four space diagonals pass through the same point, bisecting each other. let be the origin. ① Identifying properties of space diagonals

The space diagonal will have length

$\sqrt{6^2+8^2+10^2} = 10\sqrt{2}$ ① calculating space diagonal

$\therefore$ bisect,

$OA = OB = 5\sqrt{2}$.

A sphere with radius $5\sqrt{2}$ can be constructed such that it passes all eight vertices, thus being the greatest length. Any points on the surface will be $< 5\sqrt{2}$. ① Explanation involving clear understanding of the sphere.

$\therefore |z| \le 5\sqrt{2}$.

# 12 continued

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(d)

$|\vec{AB}| = \sqrt{(-b)^2 + (-b)^2 + a^2} = \sqrt{a^2+2b^2}$

side length of base is 2b.

$2(2b) = \sqrt{a^2+2b^2}$

$4b = \sqrt{a^2+2b^2}$

$16b^2 = a^2+2b^2$

$a^2 = 14b^2$

$b^2 = \frac{a^2}{14}$

$b = \frac{a}{\sqrt{14}}$

① substantial working leading to answer

① correct answer.

(e) multiply through by $-i$.

$z^2 + 2iz\sqrt{2} + 2\sqrt{3}i = 0$

$(z+i\sqrt{2})^2 - (i\sqrt{2})^2 + 2\sqrt{3}i = 0$

$(z+i\sqrt{2})^2 + 2 + 2\sqrt{3}i = 0$

$(z+i\sqrt{2})^2 = -2-2i\sqrt{3}$

① simply question.

$z+i\sqrt{2} = \pm\sqrt{-2-2\sqrt{3}i}$

$z+i\sqrt{2} = \pm\sqrt{(-1)^2+(\sqrt{3}i)^2+2\times(-1)(\sqrt{3}i)}$

inspection saves the need to do $(x+iy)^2 = -2-2\sqrt{3}i$

$z+i\sqrt{2} = \pm\sqrt{(\sqrt{3}i-1)^2}$

$z+i\sqrt{2} = \pm(\sqrt{3}i-1)$

$\therefore z+i\sqrt{2} = \sqrt{3}i-1$ or $z+i\sqrt{2} = 1-\sqrt{3}i$.

② inspection method.

or

② $(x+iy)^2$ method or anything else.

$\therefore z = -1+(\sqrt{3}-\sqrt{2})i$ or $z = 1-(\sqrt{3}+\sqrt{2})i$.

① correct answers.

# 13

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Question 13

(a) Required to prove by a contrapositive statement

"if $x^2$ is divisible by 9 then $x$ is a multiple of 3"

suppose $x$ is not a multiple of 3;

let $x = 3k + j$, where $k \in \mathbb{Z}$ and $j = 1, 2$.

$x^2 = (3k+j)^2$

$= 9k^2 + 6k + j^2$

$x^2 = 3(3k^2 + 2k) + j^2$

now, since $j = 1$ or $2$ then

$x^2 = 3p + 1$ or $3p + 4$, where $p = 3k^2 + 2k$

which are not multiples of 9.

If $x$ is not a multiple of 3, then $x^2$ is not a multiple of 9

$\therefore$ If $x^2$ is a multiple of 9, then $x$ is a multiple of 3 by contrapositive.

Conversely, if $x$ is a multiple of 3, let $x = 3j$ for integer $j$.

$x^2 = (3j)^2$

$= 9j^2$

$= 9p$ where $p = j^2$

$\therefore x^2$ is divisible by 9.

$\therefore x^2$ is divisible by 9 if and only if $x$ is a multiple of 3.

(2) show for both cases.

1. $x \neq 3j$

2. $x = 3j$

(b) let $x = -6$

$|2(-6)+5| = 7 \leq 9$

but

$|-6| = 6 > 4$

so statement is false.

(1) possible substitution for $x$ to disprove statement)

(1) Correct mathematical reasoning.

(c) M1/

LHS - RHS

$= 2^{-a^2} - 2^{-b^2}$

$= \dfrac{1}{2^{a^2}} - \dfrac{1}{2^{b^2}}$

$= \dfrac{2^{b^2} - 2^{a^2}}{2^{a^2+b^2}}$

now, $2^{b^2} - 2^{a^2} < 0$ since $a > b > 0$

and, $2^{a^2+b^2} > 0$

$\therefore \dfrac{2^{b^2} - 2^{a^2}}{2^{a^2+b^2}} < 0$

$\therefore 2^{-a^2} < 2^{-b^2}$

M2/ $a > b$

$a^2 > b^2$ since $a > b > 0$

$2^{-a^2} < 2^{-b^2}$ since $2^{-x}$ is decreasing function

(2) Correct method and mathematical reasoning.

# 13 continued

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(d)

let $P(n)$ represent the proposition.

RTP for $n \geq 2$

so

① $P(2)$ is true since.

$$1^3 < \frac{2^4}{4} < 1^3 + 2^3$$

or

$$1 < 4 < 9$$

② Assume true for $P(k)$, for some arbitrary $k \geq 2$.

then;

$$1^3 + 2^3 + \dots + (k-1)^3 < \frac{k^4}{4} < 1^3 + 2^3 + \dots + k^3$$

③ RTP $P(k+1)$ $1^3 + 2^3 + \dots + k^3 < \frac{(k+1)^4}{4} < 1^3 + 2^3 + \dots + (k+1)^3$

From $P(k)$

$$1^3 + 2^3 + \dots + (k-1)^3 < \frac{k^4}{4} < 1^3 + 2^3 + \dots + k^3$$

adding $k^3$.

$$1^3 + 2^3 + \dots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 < 1 + 2^3 + \dots + k^3 + k^3$$

we can now ignore the left hand part as we begin to add values to middle and right parts.

$$\frac{k^4}{4} + k^3 < 1 + 2^3 + \dots + k^3 + k^3$$

what we require for the middle term is $\frac{(k+1)^4}{4}$ (expanded is $\frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{4}$.

so; adding $\frac{3}{2}k^2$, $k$ and $\frac{1}{4}$ to both sides.

$$\frac{k^4}{4} + k^3 + \frac{3}{2}k^2 + k + \frac{1}{4} < 1^3 + 2^3 + \dots + k^3 + k^3 + \frac{3}{2}k^2 + k + \frac{1}{4}.$$

$$\frac{(k+1)^4}{4} < 1^3 + 2^3 + \dots + k^3 + k^3 + \frac{3}{2}k^2 + k + \frac{1}{4}$$

what we require on the RHS is $(k+1)^3$ expanded is $k^3 + 3k^2 + 3k + 1$.

now; $k^3 + \frac{3}{2}k^2 + k + \frac{1}{4} < k^3 + 3k^2 + 3k + 1$

$$\therefore \frac{(k+1)^4}{4} < 1^3 + 2^3 + \dots + k^3 + (k+1)^3$$

$$\therefore 1^3 + 2^3 + \dots + k^3 < \frac{(k+1)^4}{4} < 1^3 + 2^3 + \dots + k^3 + (k+1)^3$$ as required.

④ $P(n)$ is true by induction.

① steps 1 and 2

③ correct mathematical reasoning on part 3.

↳ 2m substantial working towards answer

1m if attempted.

(e) Consider $f(x) = (x + x^3 + x^5)(1 + x^2 + x^4)$.

$-f(x) = -(x + x^3 + x^5)(1 + x^2 + x^4)$.

$f(-x) = (-x - x^3 - x^5)(1 + x^2 + x^4)$

$= -(x + x^3 + x^5)(1 + x^2 + x^4)$.

now $f(-x) = -f(x)$ $\therefore f(x)$ is odd

$\therefore \int_{-a}^{a} f(x)\,dx$ for $f(x)$ is odd $= 0$.

$\therefore \int_{-2}^{2} (x + x^3 + x^5)(1 + x^2 + x^4)\,dx = 0$.

① recognising odd function or ② expanding and integrating.

① correct answer

(f) $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$.

$$I = \int \frac{4\sin^2\theta}{\sqrt{4 - 4\sin^2\theta}} \times 2\cos\theta\,d\theta$$

$$= \int \frac{4\sin^2\theta}{2\cos\theta} \times 2\cos\theta\,d\theta$$

$$= 4\int \frac{1}{2}(1 - \cos 2\theta)\,d\theta$$

$$= 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= 2\theta - 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4 - x^2}}{2} + C.$$

① Trig substitution

② correct working out leading to correct solution.

# 14

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Question (14)

(a) All diagonals pass through (1,1,1) plus one of the four base vertices, $A(0,0,0)$, $B(2,0,0)$, $C(2,2,0)$ and $D(0,2,0)$.

$$\underset{\sim}{r}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1-0\\1-0\\1-0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\underset{\sim}{r}_2 = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1-2\\1-0\\1-0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\underset{\sim}{r}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1-2\\1-2\\1-0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

$$\underset{\sim}{r}_4 = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1-0\\1-2\\1-0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

(2) All correct equation

(2) Analysis of direction vectors and dot product.

Consider direction vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}$

none of the dot products of direction vectors result in $0$, so, not perpendicular.

(b) midpoints

$M_{AB} = (0,1,2)$

$M_{BC} = (2,2,2)$

$M_{AC} = (2,1,0)$

The equations through each vertex are.

$$\underset{\sim}{r}_A = \begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2-0\\2-0\\2-0\end{pmatrix} = \begin{pmatrix}2\lambda\\2\lambda\\2\lambda\end{pmatrix} \quad \ldots (1)$$

$$\underset{\sim}{r}_B = \begin{pmatrix}0\\2\\4\end{pmatrix} + \mu\begin{pmatrix}2-0\\1-2\\0-4\end{pmatrix} = \begin{pmatrix}2\mu\\2-\mu\\4-4\mu\end{pmatrix} \quad \ldots (2)$$

$$\underset{\sim}{r}_C = \begin{pmatrix}4\\2\\0\end{pmatrix} + \phi\begin{pmatrix}0-4\\1-2\\2-0\end{pmatrix} = \begin{pmatrix}4-4\phi\\2-\phi\\2\phi\end{pmatrix} \quad \ldots (3)$$

(1) All 3 equations

For (1) and (2)

$2\lambda = 2\mu$ $\qquad$ $2\lambda = 2-\mu$ $\qquad$ $2\lambda = 4-4\mu$

$\lambda = \mu$ $\qquad$ $3\lambda = 2$

$\lambda = \frac{2}{3}$

$\therefore \mu = \frac{2}{3}$

when $\lambda = \mu = \frac{2}{3}$

$\underset{\sim}{r}_A = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$

$\underset{\sim}{r}_B = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$ $\qquad$ $\therefore$ intersects at $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$.

now (1) and (3)

taking $z$ coordinate.

$2\lambda = 2\phi$

$\lambda = \phi$

now $y$ coordinate.

$2\lambda = 2-\phi$

$3\lambda = 2$

$\lambda = \frac{2}{3}$

(2) correct point of intersection and working out leading to answer.

when $\lambda = \phi = \frac{2}{3}$.

$\underset{\sim}{r}_A = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$

$\underset{\sim}{r}_C = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$. $\qquad$ $\therefore$ intersects at $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$

Since (2) and (3) intersect (1) at $\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$ then (2) and (3) also share same intersection point.

Thus, medians intersect at one point (concurrent).

# 14 continued

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(c) i.


Force = ma = sum of all forces. ① Definition.

$$\therefore \; 20\ddot{x} = 100 - kv^2$$

$$\ddot{x} = 5 - \frac{kv^2}{20}$$ where $\ddot{x} = a$ and k is a constant.

① Proof of statement.

ii. $$\ddot{x} = 5 - \frac{kv^2}{20}$$

$$v\frac{dv}{dx} = \frac{100 - kv^2}{20}$$

$$\frac{dx}{dv} = \frac{20v}{100 - kv^2}.$$

① Correct/potential
- substitution for 'a' or '$\ddot{x}$' (not limited to $v\frac{dv}{dx}$).
- Initial / final conditions

Recognises separable nature of DE.

$$x = \int_0^v \frac{20v}{100 - kv^2}\,dv$$

$$= -\frac{10}{k}\Big[\ln(100 - kv^2)\Big]_0^v$$

$$x = -\frac{10}{k}\ln\frac{100}{100 - kv^2}$$

① Correct answer.

(d) $$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

Easiest to start off by proving GM ≤ AM (should be directly taught in course).

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

Start with fact:

$$(\sqrt{a} - \sqrt{b})^2 \geq 0$$

$$a + b \geq 2\sqrt{ab}.$$

$$\sqrt{ab} \leq \frac{a+b}{2}$$ QED.

① acceptable proof for GM ≤ AM

now,

$$\sqrt{ab} \leq \frac{a+b}{2} \qquad (\times\sqrt{ab})$$

$$ab \leq \frac{(a+b)\sqrt{ab}}{2}$$

$$\frac{2ab}{a+b} \leq \sqrt{ab}$$

① Acceptable proof for HM ≤ GM.

now;

$$\left(\frac{a}{2} - \frac{b}{2}\right)^2 \geq 0$$

$$\frac{a^2 - 2ab + b^2}{4} \geq 0$$

$$\frac{a^2 - 2ab + b^2}{4} + \frac{(a+b)^2}{4} \geq \frac{(a+b)^2}{4}$$

$$\frac{2a^2 + 2b^2}{4} \geq \frac{(a+b)^2}{4}$$

$$\frac{(a+b)^2}{4} \leq \frac{a^2+b^2}{4}.$$

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

or alternative AM ≤ QM.

RTP: $\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a^2+b^2}{2}\right)$$

$$= \frac{a^2 + 2ab + b^2}{4} - \frac{a^2+b^2}{2}$$

$$= -\left(\frac{a-b}{2}\right)^2$$

$$\leq 0$$

(Considering the difference).

① Acceptable proof for AM ≤ QM.

Putting them together.

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

① Putting everything together.

# 15

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Question 15

(a) i. $x = vt\cos\theta$

$t = \dfrac{x}{v\cos\theta}$

$y = -\frac{1}{2}g\left(\dfrac{x}{v\cos\theta}\right)^2 + v\left(\dfrac{x}{v\cos\theta}\right)\sin\theta$

$= -\dfrac{gx^2}{2v^2\cos^2\theta} + \dfrac{x\sin\theta}{\cos\theta}$

$y = x\tan\theta - \dfrac{gx^2}{2v^2}\sec^2\theta.$ ① correct mathematical proof

ii. let P be $\left(\frac{D}{\sqrt{2}}, -\frac{D}{\sqrt{2}}\right)$

$\because$ lies on line $y = -x$ (indicated by $\frac{\pi}{4}$ angle)

and $OP = D$ (given). ① recognises point P

$\therefore -\dfrac{D}{\sqrt{2}} = \dfrac{D}{\sqrt{2}}\tan\theta - \dfrac{g\left(\frac{D}{\sqrt{2}}\right)^2}{2v^2}\sec^2\theta$

$-\dfrac{D}{\sqrt{2}} = \dfrac{D}{\sqrt{2}}\tan\theta - \dfrac{gD^2}{4v^2}\sec^2\theta$ $(\times\cos^2\theta)$.

$-\dfrac{D}{\sqrt{2}}\cos^2\theta = \dfrac{D}{\sqrt{2}}\sin\theta\cos\theta - \dfrac{gD^2}{4v^2}$ ① substantial working leading to solution.

$\dfrac{gD^2}{4v^2} - \dfrac{D}{\sqrt{2}}(\sin\theta\cos\theta + \cos^2\theta) = 0$

$D\left(\dfrac{gD}{4v^2} - \dfrac{1}{\sqrt{2}}\cos\theta(\sin\theta + \cos\theta)\right) = 0$

$\therefore D = 0$ or $\dfrac{gD}{4v^2} = \dfrac{1}{\sqrt{2}}\cos\theta(\sin\theta + \cos\theta)$

$D = 2\sqrt{2}\dfrac{v^2}{g}\cos\theta(\sin\theta + \cos\theta)$ ① correct proof.

iii. $\dfrac{dD}{d\theta} = 2\sqrt{2}\dfrac{v^2}{g}\left[(\cos\theta + \sin\theta)(-\sin\theta) + (\cos\theta)(-\sin\theta + \cos\theta)\right]$

$= 2\sqrt{2}\dfrac{v^2}{g}\left[-\cos\theta\sin\theta - \sin^2\theta - \sin\theta\cos\theta + \cos^2\theta\right]$ ① Differentiation

$= 2\sqrt{2}\dfrac{v^2}{g}\left[\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta\right]$

$= 2\sqrt{2}\dfrac{v^2}{g}(\cos 2\theta - \sin 2\theta).$ ① correct proof

iv. $\dfrac{d^2D}{d\theta^2} = 2\sqrt{2}\dfrac{v^2}{g}\left[-2\sin 2\theta - 2\cos 2\theta\right]$

$= -4\sqrt{2}\dfrac{v^2}{g}(\sin 2\theta + \cos 2\theta)$ for $0 \le \theta \le \frac{\pi}{2}$

$\therefore$ D has max value since $2^{nd}$ derivative $< 0$. ① showing maximum turning pt. (either by second derivative or otherwise).

let $\dfrac{dD}{d\theta} = 0$ to find turning pt.

$2\sqrt{2}\dfrac{v^2}{g}(\cos 2\theta - \sin 2\theta) = 0$

$\cos 2\theta - \sin 2\theta = 0$

$\cos 2\theta = \sin 2\theta$

$\tan 2\theta = 1$

$2\theta = \frac{\pi}{4}$

$\theta = \frac{\pi}{8}$

$\therefore$ Max D at $\theta = \frac{\pi}{8}$. ① correct answer.

# 15 continued

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(b) i. M1/

Consider

$I_n + I_{n-2}$

$= \int_0^{\frac{\pi}{4}} \tan^n x \, dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\tan^2 x + 1) \, dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx$

$= \int_0^{\frac{\pi}{4}} \frac{d}{dx}\left(\frac{\tan^{n-1} x}{n-1}\right) dx$

$= \frac{1}{n-1}\left[\tan^{n-1} x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{n-1}$

$\therefore I_n + I_{n-2} = \frac{1}{n-1}$

$\therefore I_n = \frac{1}{n-1} - I_{n-2}$ QED.

① correct consideration $I_n + I_{n-2}$

① Recognises reverse chain rule

① correct answer.

M2/

$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x \, dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\sec^2 x - 1) \, dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx.$

$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$

$= \frac{1}{n-1} - I_{n-2}.$

① splitting $\tan^n x$ into $\tan^{n-2} x$ and $\tan^2 x$

① correct mathematical analysis

① correct answer.

ii. $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx = I_5$

$I_5 = \frac{1}{4} - I_3$

$I_3 = \frac{1}{2} - I_1$

$I_1 = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$

$= \left[-\ln|\cos x|\right]_0^{\frac{\pi}{4}}$

$= -\ln\left|\frac{\sqrt{2}}{2}\right| - (-\ln 1)$

$= -\ln\left(\frac{\sqrt{2}}{2}\right)$

$= \ln\left(\frac{2}{\sqrt{2}}\right)$

$I_1 = \ln\sqrt{2}$

$\therefore I_5 = \frac{1}{4} - \left(\frac{1}{2} - \ln\sqrt{2}\right)$

$= -\frac{1}{4} + \ln\sqrt{2}$ or $\frac{1}{2}\ln 2 - \frac{1}{4}.$

① substantial working leading to answer.

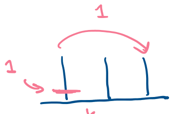
① correct answer

16

Tuesday, 15 September 2020 12:48 am

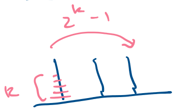
Question 16

(a) let $P(n)$ represent proposition.



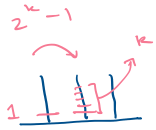

$P(1)$ is true since the one disk can be moved to the third peg in 1 move and $2^1 - 1 = 1$

① Prove true for $n=1$




Assume true for $P(k)$; it takes $2^k - 1$ moves to move k pegs to the third peg from the first peg.

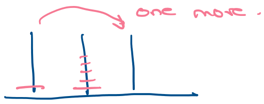
Prove for $P(k+1)$; it takes $2^{k+1} - 1$ moves to move $k+1$ pegs from first to third peg.




From $P(k)$ we can assume; instead of moving straight to third peg, we move to second peg first, in the same $2^k - 1$ moves.

③ Demonstrates strong conceptual understanding and proving the statement true.

now we move the base disc from 1 to 3.




and moving the original k pegs all to the third peg would again require $2^k - 1$ moves from proposition.

so;

total moves $= 2^k - 1 + 1 + 2^k - 1$

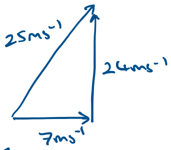
$= 2 \cdot 2^k - 1$

$= 2^{k-1} - 1$ as required.

$\therefore P(n)$ is true by induction for $n \geq 1$.

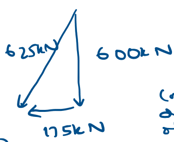
(b) i. $V = \sqrt{7^2 + 24^2} = 25\ ms^{-1}$

$\therefore R = -k \times 25^2 = -625k$ N.




Now triangle representing velocity and components and resistance and components must be similar since resistance opposes the motion in opposite direction and equal to some proportion of its magnitude. hence when put together will form a triangle with same angles as original velocity triangle (equiangular).

① recognises similar triangles




(notice direction of arrows).

① correct components for resistance

by scaling; $1 : -25k$

horizontal component is $-175k$ and vertical component is $-600k$ N.

① Prove satisfies Pythagoras'.

Using magnitudes of resistance and components.

$175^2 + 600^2 = 390625 = 625^2$ (ignoring k).

$\therefore$ satisfies Pythagoras' theorem.

ii. $R_x = -kv\dot{x} = -k \times 25 \times 7 = -175k$

$R_y = -kv\dot{y} = -k \times 25 \times 24 = -600k$

② Show for $R_x$
Show for $R_y$.

$\therefore$ horizontal and vertical components at point can be found using $R_x = -kv\dot{x}$ and $R_y = -kv\dot{y}$

# 16 continued

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(c) i. $m\ddot{x} = -mg - \frac{mv}{10}$

$\therefore \ddot{x} = -g - \frac{v}{10}$

$\frac{dv}{dt} = -g - \frac{v}{10}$

① correct answer.

ii. $\frac{dt}{dv} = \frac{-10}{10g+v}$

$t = -\int_{10(20-g)}^{v} \frac{10}{10g+v}\, dv$

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

$= 10\left[\ln(10g+v)\right]_v^{10(20-g)}$

① correct integration w/ initial and final conditions.

$= 10\left[\ln(10g + 200 - 10g) - \ln(10g+v)\right]$

$t = 10\ln\left(\frac{200}{10g+v}\right)$

when $t = T$, $v = 0$.

$T = 10\ln\left(\frac{20}{g}\right)$ as required.

① correct proof.

iii. $\frac{dt}{dv} = -\frac{10}{10g+v}$

... same as part (ii)

$t = 10\ln\left(\frac{200}{10g+v}\right)$

$\frac{t}{10} = \ln\left(\frac{200}{10g+v}\right)$

$e^{\frac{t}{10}} = \frac{200}{10g+v}$

$10ge^{\frac{t}{10}} + ve^{\frac{t}{10}} = 200$

$v = \frac{200 - 10ge^{\frac{t}{10}}}{e^{\frac{t}{10}}}$

① correct mathematical analysis leading to answer.

$= 200e^{-\frac{t}{10}} - 10g$

$\therefore H = \int_0^T 200e^{-\frac{t}{10}} - 10g\; dt$

$= \left[-2000e^{-\frac{t}{10}} - 10gt\right]_0^T$

$= \left(-2000e^{-\frac{T}{10}} - 10gT\right) - (-2000 - 0)$

$= 2000 - 2000e^{-\ln\left(\frac{20}{g}\right)} - 10gT$

$= 2000 - 100g - 10gT$

① correct proof.

$H = 2000 - 10g[10+T]$ as required.

iv. $\ddot{x} = -g - \frac{v}{10}$

$0 = g - \frac{V_T}{10}$

$V_T = 10g$

① correct answer.